PSEUDO-SYMMETRIC LIE GROUPS OF DIMENSION 3

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Abstract. Pseudo-symmetry of 3-dimensional Lie groups is investigated.

Introduction

A Riemannian 3-manifold (M, g) is said to be a proper *pseudo-symmetric space* if its principal Ricci curvatures (also called *Ricci eigenvalues*) $\{\rho_1, \rho_2, \rho_3\}$ satisfy the relation $\rho_1 = \rho_2 \neq \rho_3$ ($\rho_3 \neq 0$) up to numeration. In particular, a proper pseudo-symmetric 3-space (M, g) is said to be of *constant type* if ρ_3 is a nonzero constant.

Such spaces have been studied by some different motivations. In particular, O. Kowalski explained some motivations of the study of pseudo-symmetric 3-spaces with *constant* principal Ricci curvatures in [7]. Kowalski and S. Ž. Nikčević [8] found the necessary and sufficient conditions for three constants ρ_1, ρ_2 and ρ_3 to be the principal Ricci curvatures of some 3-dimensional locally homogeneous Riemannian manifolds.

In this short note, we give a list of 3-dimensional Lie groups with left invariant metric which are pseudo-symmetric.

1. Preliminaries

Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and a left invariant Riemannian metric $\langle\cdot,\cdot\rangle$. Then the *Levi-Civita connection* ∇ of $(G, \langle\cdot,\cdot\rangle)$ is described by the *Koszul formula*:

$$2\langle\nabla_X Y, Z\rangle = -\langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle, \quad X, Y, Z \in \mathfrak{g}.$$

Let us define a symmetric bilinear map $U : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle \tag{1}$$

and call it the *natural-reducibility obstruction* of $(G, \langle\cdot,\cdot\rangle)$. One can see that the metric g is right-invariant if and only if $U = 0$.

A Lie group G is said to be *unimodular* if its left invariant Haar measure is right invariant. J. Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional oriented Lie algebra with an inner product $\langle\cdot,\cdot\rangle$. Denote by $\times$ the *vector product operation* of the oriented inner product space $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. The vector product operation is a skew-symmetric bilinear map $\times : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is uniquely determined by the following conditions:

(i) $\langle X, X \times Y\rangle = \langle Y, X \times Y\rangle = 0$,
(ii) $|X \times Y|^2 = \langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2$,
(iii) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$,

Revised version of the article published in;
Bulletin of Faculty of Education, Utsunomiya University, Section 2, **57**, (2007), 1–5.
This works is partially supported by Grant-in-Aid for Encouragement of Young Researchers, Utsunomiya University (2006).

for all $X, Y \in \mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X, Y] = L_{\mathfrak{g}}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G, the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined. The unimodularity of G is characterised as follows.

Proposition 1. ([10]) *Let G be an oriented 3-dimensional Lie group with a left invariant Riemannian metric. Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

2. Unimodular Lie groups

Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot,\cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \qquad c_i \in \mathbb{R}.$$

Three-dimensional unimodular Lie groups are classified by Milnor as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+,+,+)$	$\mathrm{SU}(2)$	compact and simple
$(+,+,-)$	$\widetilde{\mathrm{SL}}_2\mathbb{R}$	non-compact and simple
$(+,+,0)$	$\widetilde{E}(2)$	solvable
$(+,-,0)$	$E(1,1)$	solvable
$(+,0,0)$	Heisenberg group Nil_3	nilpotent
$(0,0,0)$	$(\mathbb{R}^3, +)$	Abelian

To describe the Levi-Civita connection ∇ of G, we introduce the following constants:

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i.$$

Proposition 2. *The Levi-Civita connection is given by*

$$\begin{array}{ccc} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \mu_1 e_3, & \nabla_{e_1} e_3 = -\mu_1 e_2 \\ \nabla_{e_2} e_1 = -\mu_2 e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = \mu_2 e_1 \\ \nabla_{e_3} e_1 = \mu_3 e_2, & \nabla_{e_3} e_2 = -\mu_3 e_1 & \nabla_{e_3} e_3 = 0. \end{array}$$

The Riemannian curvature R is given by

$$R(e_1, e_2)e_1 = (\mu_1\mu_2 - c_3\mu_3)e_2, \quad R(e_1, e_2)e_2 = -(\mu_1\mu_2 - c_3\mu_3)e_1,$$

$$R(e_2, e_3)e_2 = (\mu_2\mu_3 - c_1\mu_1)e_3, \quad R(e_2, e_3)e_3 = -(\mu_2\mu_3 - c_1\mu_1)e_2,$$

$$R(e_1, e_3)e_1 = (\mu_3\mu_1 - c_2\mu_2)e_3, \quad R(e_1, e_3)e_3 = -(\mu_3\mu_1 - c_2\mu_2)e_1.$$

The basis $\{e_1, e_2, e_3\}$ diagonalises the Ricci tensor. The principal Ricci curvatures are given by

$$\rho_1 = 2\mu_2\mu_3, \quad \rho_2 = 2\mu_1\mu_3, \quad \rho_3 = 2\mu_1\mu_2.$$

The natural-reducibility obstruction U is given by

$$U(e_1, e_2) = \frac{1}{2}(-c_1 + c_2)e_3, \quad U(e_1, e_3) = \frac{1}{2}(c_1 - c_3)e_2, \quad U(e_2, e_3) = \frac{1}{2}(-c_2 + c_3)e_1.$$

2.1. $G = \mathrm{SU}(2)$**.** Without loss of generality, we may assume that $c_1 \geq c_2 \geq c_3 > 0$. Under this normalisation we have the following result.

Proposition 3. *The special unitary group* $\mathrm{SU}(2)$ *is pseudo-symmetric if and only if the structure constants* $\{c_1, c_2, c_3\}$ *satisfies one of the following*:

(1) $c_1 = c_2$. *In this case,* $\rho_1 = \rho_2 = c_3(2c_1 - c_3)/2$, $\rho_3 = c_3^2/2$.
(2) $c_2 = c_3$. *In this case,* $\rho_1 = c_1^2/2$, $\rho_2 = \rho_3 = c_1(-c_1 + 2c_2)/2$.
(3) $c_1 = c_2 + c_3$. *In this case* $\rho_1 = c_2c_3/2$, $\rho_2 = \rho_3 = 0$.

In particular, $\mathrm{SU}(2)$ *is locally symmetric if and only if* $c_1 = c_2 = c_3 = c$. *In such a case, the metric is bi-invariant and of constant positive curvature* $4/c^2$.

Note that the dimension $\dim I(\mathrm{SU}(2))$ of the isometry group of $\mathrm{SU}(2)$ is greater than 3 if and only if the metric is pseudo-symmetric.

2.2. $G = \widetilde{\mathrm{SL}}_2\mathbb{R}$**.** In this case, the space of left invariant metrics is parametrised by the structure constants with condition $c_1 \geq c_2 > 0 > c_3$. Under this parametrisation we have $\mu_1 < 0$ and $\mu_3 > 0$. The following discussions were done by Tsukada [13].

(1) $\mu_2 > 0$: In this case, the principal Ricci curvatures are mutually distinct.
(2) $\mu_2 = 0$: In this case $\rho_1 = \rho_3 = 0$ and $\rho_2 < 0$.
(3) $\mu_2 < 0$: If $c_1 > c_2$ then the principal Ricci curvatures are mutually distinct. If $c_1 = c_2$, then $\rho_1 = \rho_2 < 0$ and $\rho_3 > 0$.

Proposition 4. *The Lie group* $\widetilde{\mathrm{SL}}_2\mathbb{R}$ *is pseudo-symmetric if and only if* (1) $c_1 = c_2 - c_3$ *or* (2) $c_1 = c_2$.

Note that $\dim I(\widetilde{\mathrm{SL}}_2\mathbb{R}) = 4$ if and only if $c_1 = c_2$.

2.3. $G = \widetilde{E}(2)$ **or** $E(1,1)$**.** If $c_3 = 0$, we have

$$\rho_1 = -\rho_2 = \frac{1}{2}(c_1^2 - c_2^2), \ \rho_3 = -\frac{1}{2}(c_1 - c_2)^2.$$

from these equations, one can deduce the following two results (*cf.* [2]):

Proposition 5. *The Lie group* $\widetilde{E}(2)$ *is pseudo-symmetric if and only if* $c_1 = c_2$. *In this case,* $\widetilde{E}(2)$ *is isometric to Euclidean* 3*-space* $\mathbb{E}^3$ (*and hence symmetric*).

Proposition 6. *The Minkowski motion group* $E(1,1)$ *is pseudo-symmetric if and only if* $c_1 = -c_2$. *In this case,* $E(1,1)$ *is isometric to the model space* Sol_3 *of* 3*-dimensional solvegeometry in the sense of Thurston* [12].

Note that Sol_3 is a proper Riemannian 4-symmetric space.

2.4. $G = \mathrm{Nil}_3$**.** . If G is the Heisenberg group, the principal Ricci curvatures are given by $\rho_1 = c_1^2/4 > 0$ and $\rho_2 = \rho_3 = -c_1^2/4$.

Proposition 7. *The Heisenberg group is a proper pseudo-symmetric space with respect to any left invariant metric.*

Remark 1. Kowalski and Nikčević obtained the following criterion for three real constants to be the principal Ricci curvatures of some 3-dimensional unimodular Lie groups.

Theorem 1. ([8, Theorem 3.1]) *Let* ρ_1, ρ_2 *and* ρ_3 *be real constants. Then a unimodular Lie group with a left invariant metric and with prescribed principal Ricci curvatures* ρ_1, ρ_2 *and* ρ_3 *exists if and only if* $\rho_1\rho_2\rho_3 > 0$ *or if at least two of* ρ_i, $i = 1, 2, 3$ *are zero.*

3. Non-unimodular Lie groups

Let G be a non-unimodular 3-dimensional Lie group with a left invariant metric. Then the *unimodular kernel* $\mathfrak{u}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{u} = \{X \in \mathfrak{g} \mid \operatorname{tr} \operatorname{ad}(X) = 0\}.$$

Here $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is a homomorphism defined by

$$\operatorname{ad}(X)Y = [X, Y].$$

One can see that $\mathfrak{u}$ is an ideal of $\mathfrak{g}$ which contains the ideal $[\mathfrak{g}, \mathfrak{g}]$.

On $\mathfrak{g}$, we can take an orthonormal basis $\{e_1, e_2, e_3\}$ such that

(1) $\langle e_1, X\rangle = 0,\ X \in \mathfrak{u}$,
(2) $\langle [e_1, e_2], [e_1, e_3]\rangle = 0$.

Then the commutation relations of the basis are given by

$$[e_1, e_2] = \alpha e_2 + \beta e_3,\ \ [e_2, e_3] = 0,\ \ [e_1, e_3] = \gamma e_2 + \delta e_3,$$

with $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. Under a suitable homothetic change of the metric, we may assume that $\alpha + \delta = 2$. Then the constants α, β, γ and δ are represented as

$$\alpha = 1 + \xi,\ \ \beta = (1 + \xi)\eta,\ \ \gamma = -(1 - \xi)\eta,\ \ \delta = 1 - \xi,$$

where (ξ, η) satisfies the condition $\xi, \eta \geq 0$ and $\xi^2 + \eta^2 \neq 0$. Under this normalisation, the Levi-Civita connection of G is given by the following table:

Proposition 8.

$$\begin{array}{lll}
\nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \eta e_3, & \nabla_{e_1} e_3 = -\eta e_2 \\
\nabla_{e_2} e_1 = -(1+\xi)e_2 - \xi\eta e_3, & \nabla_{e_2} e_2 = (1+\xi)e_1, & \nabla_{e_2} e_3 = \xi\eta e_1 \\
\nabla_{e_3} e_1 = -\xi\eta e_2 - (1-\xi)e_3, & \nabla_{e_3} e_2 = \xi\eta e_1 & \nabla_{e_3} e_3 = (1-\xi)e_1.
\end{array}$$

The Riemannian curvature R is given by

$$\begin{aligned}
R(e_1, e_2)e_1 &= \{\xi\eta^2 + (1+\xi)^2 + \xi\eta^2(1+\xi)\}e_2, \\
R(e_1, e_2)e_2 &= -\{\xi\eta^2 + (1+\xi)^2 + \xi\eta^2(1+\xi)\}e_1, \\
R(e_1, e_3)e_1 &= -\{\xi\eta^2 - (1-\xi)^2 + \xi\eta^2(1-\xi)\}e_3, \\
R(e_1, e_3)e_3 &= \{\xi\eta^2 - (1-\xi)^2 + \xi\eta^2(1-\xi)\}e_1, \\
R(e_2, e_3)e_2 &= \{1 - \xi^2(1+\eta^2)\}e_3, \\
R(e_2, e_3)e_3 &= -\{1 - \xi^2(1+\eta^2)\}e_2.
\end{aligned}$$

The basis $\{e_1, e_2, e_3\}$ diagonalises the Ricci tensor. The principal Ricci curvatures are given by

$$\rho_1 = -2\{1 + \xi^2(1+\eta^2)\},\quad \rho_2 = -2\{1 + \xi(1+\eta^2)\},\quad \rho_3 = -2\{1 - \xi(1+\eta^2)\}.$$

The natural-reducibility obstruction U is given by

$$U(e_1, e_2) = -\frac{1}{2}\{(1+\xi)e_2 + (1-\xi)\eta\, e_3\},\quad U(e_1, e_3) = -\frac{\eta}{2}\{(1+\xi)e_2 + (1-\xi)e_3\},$$

$$U(e_2, e_2) = (1+\xi)e_1,\quad U(e_2, e_3) = -\xi\eta\, e_1,\quad U(e_3, e_3) = -(1-\xi)e_1.$$

The Lie algebra $\mathfrak{g}$ is classified by the Milnor's invariant $\mathcal{D} = (1-\xi^2)/(1+\eta^2)$.

Direct computation shows the following result.

Proposition 9. *A non-unimodular Lie group G is pseudo-symmetric if and only if $\xi = 0$ or $\xi = 1$. In particular, G is semi-symmetric if and only if $\xi = 0$ or $(\xi, \eta) = (1, 0)$. In the former case ($\xi = 0$), G is of constant curvature -1. In the latter case ($\xi = 1, \eta = 0$), G is isometric to the direct product $\mathbb{E}^1 \times H^2(-1)$. Hence G is semi-symmetric if and only if G is locally symmetric.*

Remark 2. An explicit matrix group model for the non-unimodular Lie group G with $\eta = 0$ is given in [5]–[6].

Remark 3. Non-unimodular Lie groups with $\xi = 1$ have sectional curvatures

$$\langle R(e_1, e_2)e_2, e_1\rangle = -4 - 3\eta^2, \quad \langle R(e_2, e_3)e_3, e_2\rangle = \langle R(e_1, e_3)e_3, e_1\rangle = \eta^2.$$

Hence the simply connected non-unimodular Lie group G with $\xi = 1$ and $\eta \neq 0$ is isometric to $\widetilde{\mathrm{SL}}_2\mathbb{R}$ with $c_1 = c_2$.

The following characterisation is due to Kowalski and Nikčević.

Theorem 2. ([8, Theorem 4.1]) *A non-unimodular Lie group with left invariant metric and prescribed principal Ricci curvatures ρ_1, ρ_2 and ρ_3 exists if and only if, either $\rho_1 = \rho_2 = \rho_3 < 0$ or, up to a possible re-numeration,*

$$2\rho_1 < \rho_2 + \rho_3 < 0, \tag{2}$$

$$\rho_1(\rho_2 + \rho_3) \leq \rho_2^2 + \rho_3^2, \tag{3}$$

$$\rho_1 - \rho_2 = k(\rho_2 - \rho_3) \quad \textit{for some constant } k. \tag{4}$$

One can see that every 3-dimensional non-unimodular Lie group with $\xi = 1$ satisfies (2), (3) and (4) with $k = 0$.

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